

# Double Clustering and Graph Navigability

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## Abstract

Graphs are called *navigable* if one can find short paths through them using only local knowledge. It has been shown that for a graph to be navigable, its construction needs to meet strict criteria. Since such graphs nevertheless seem to appear in nature, it is of interest to understand why these criteria should be fulfilled.

In this paper we present a simple method for constructing graphs based on a model where nodes vertices are “similar” in two different ways, and tend to connect to those most similar to them - or cluster - with respect to both. We prove that this leads to navigable networks for several cases, and hypothesize that it also holds in great generality. Enough generality, perhaps, to explain the occurrence of navigable networks in nature.

## 1 Introduction

Motivated by the small-world experiments of Stanley Milgram [16], and the models for social networks inspired by them [20], Jon Kleinberg introduced the question of whether graphs can be searched (or navigated) in a decentralized manner [13]. In particular, he showed that when a grid structure is augmented by random edges, whether it is possible to use those edges to efficiently route queries from one point in the grid to another depends on their distribution. In particular, if each vertex  $x$  in a  $d$ -dimensional grid is given one additional “long range” link to some vertex, beyond those to its nearest neighbors, then when the probability of  $y$  being the selected is proportional to  $|x - y|^{-d}$ , any greedy walk on the resulting graph is expected to complete in a number of steps polylogarithmic to the graph size. If the probability of  $y$  being selected is any other exponent of the distance (in particular 0, meaning the long-range link is uniformly selected) then any form of routing

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which uses only information about the points seen thus far will require a number of steps which is a fractional power of the graph size.

A natural question following from Kleinberg’s results is to ask if there is any dynamic which might cause the frequency of edges in naturally occurring graphs to have the sought relationship with their length. Several empirical studies of social network data following Kleinberg have observed just such a relationship [1] [15], making it plausible that such a dynamic may exist, but it has not been identified.

In this paper we observe that the desired edge distribution arises naturally in another probabilistic model, that of best-yet sampling from a population, and use this to show how navigable networks may arise when vertices belong to two independent spaces and tend to cluster in both (in social network terms, these may be identified with the physical world and metaphorical space of “interests” - people tend to be befriended those who are close in either sense.) The resulting spatial random graph, which we dub the double clustering graph, turns out to be navigable with respect to both spaces.

## 1.1 Previous Work

The original questions about navigability in a geographical setting were posed and answered by Kleinberg in [13] and [12]. Later, Kleinberg [14] and Watts et al. [19] independently proposed similar models based on the categorization of ideas or characteristics. The latter paper includes the idea that vertices may be similar in several independent spaces, but does not discuss how this might lead to the desired edge distribution. Fraigniaud [9] went further and discussed augmentation in more general settings based on tree-decompositions of general graphs.

Some conceptually different work has been done previously to try to explain the emergence of Kleinberg type edge frequencies. In particular, [4] [18] and [17] propose graph rewiring processes which seem to create navigable networks in their stationary state. These may help explain how such networks arise under some circumstances, but are not always an easy fit with observed reality, and have so far eluded complete analysis. [8] shows a form of navigable augmentation that depends on little knowledge of the base graph, but this algorithm is complicated and does not give an intuitive reason why the desired edge distribution should arise.

## 1.2 Contribution

We characterize our contribution as follows:

- We introduce the “double clustering” graph construction. This is a simple rule for constructing a graph between a set of vertices with positions in two different spaces, so that they tend to connect to those nearest in both. Double clustering can be seen as a spatial or combinatorial construction depending on whether the points are originally placed in graphs or metric spaces.
- We show analytically that in several cases double clustering leads to navigable graphs.
- We hypothesize that this holds for a much larger class of such graphs, something we illustrate with simulation of several relevant sub-models.

## 2 Navigable Graphs

### 2.1 Decentralized Routing

Let  $G = (V, E)$  be a connected finite graph of high (some power of  $|V|$ ) diameter, and let the random graph  $G'$  be created by addition (augmentation) of random edges to  $G$ . It is well known, see for instance [3], that the diameter shrinks quickly to a logarithm of  $|V|$  when random edges are added between the vertices. Navigability concerns not a small diameter, however, but rather a stronger property: the possibility of finding a short path between two vertices in  $G'$  using only local knowledge at vertex visited. By local knowledge, one means that each vertex knows  $G$ , but does not know which random edges have been added to any vertex until it is visited. The exact limits of such *decentralized routing algorithms* have been discussed elsewhere [13] [2], but since we are interested only in upper bounds, we will define only the subset of such algorithms of interest to us. When routing from for some vertex  $z$ , in each step we will select as the next vertex in the path a  $G'$ -neighbor of the current vertex,  $x$ . This choice will be made entirely as a function of each neighbors  $G$ -distance to  $z$ , and nothing else. All such algorithms are decentralized by Kleinberg’s definition.

The most direct decentralized algorithm, and the most important, is *greedy routing*. In greedy routing, the next vertex chosen is that neighbor which is closest to  $z$  in  $G$  (with some tie-breaking rule applied). Note that both the original and augmented edges can be used, but because the choice is only optimal with respect to  $G$ , the path discovered by greedy routing will seldom be a minimal path in  $G'$ . In one case below we will modify the routing to divert from a greedy choice slightly for technical reasons, but the principle is still the same.

We start with  $G$  as a  $d$ -dimensional  $n$ -grid (that is  $V = \{1, 2, \dots, n\}^d$  and there are edges between adjacent vertices) and independently add a single

directed edge from each vertex to a random destination. The long-range connection is added such that for  $x, y \in V$ , and some  $\alpha \geq 0$

$$\mathbf{P}(x \rightsquigarrow y) = \frac{1}{h_{\alpha,n}|x - y|^\alpha} \quad (1)$$

where  $x \rightsquigarrow y$  is the event that  $x$  is augmented with an edge to  $y$  and  $|x - y|$  denotes  $L^1$  distance in  $\mathbb{Z}^d$ .  $h_{\alpha,n}$  is a normalizing constant.

The by now well known result of Kleinberg is that when  $\alpha = d$ , greedy routing between any two points in  $V$  takes  $O(\log^2 n)$  steps in expectation, while for any other value of  $\alpha$  decentralized algorithm creates routes of expected length at least  $\Omega(n^s)$  steps for some  $s > 0$  (where  $s$  depends on the dimension but not the algorithm chosen).

## 2.2 Doubling Dimension and More General Augmentation

It should be noted that if the graph is a  $d$ -dimensional grid as above, and for  $x \in V$   $B_r(x) = \{y \in V : |y - x| \leq r\}$ , then  $|B_r(x)| \propto r^d$ . For  $\alpha = d$  (1) can then be interpreted as

$$\mathbf{P}(x \rightsquigarrow y) \propto \frac{1}{|B_r(x)|}. \quad (2)$$

This general principle, that under navigable augmentation the probability that  $x$  links to  $y$  should be inversely proportional to the number of vertices that are closer to  $x$  than  $y$  has been observed to hold across a wider class of graphs than just the grids, see e.g. [14] [7] [15], and seems to be the general principle behind navigability. It leads directly to our first construction.

A natural generalization to grids is to study graphs which are naturally grid-like. Let  $B_r(v)$  be as above, but using graph instead of grid distance.

**Definition 2.1.** *A family of graphs has bounded doubling dimension if there exists a family wide constant  $c$  such that for all  $G = (V, E)$  in the family,  $u, v \in V$ , and  $r \geq 1$*

$$B_r(u) \subset B_{2r}(u) \Rightarrow |B_{2r}(v)| \leq c|B_r(u)|.$$

The commonly used *doubling dimension* of the family roughly corresponds to the  $\log_2$  of the smallest such  $c$ . This is not the widest class of graphs where navigable augmentation is possible, [7] and [10] have shown that families with a sufficiently slowly growing dimension can still be made navigable, but it provides a a good compromise between generality and convenience for us.

In the constructions below, we will augment the base graph with more than one long-range edge per vertex. In general, a  $k$  edge augmentation is expected to give  $O(\log^2 n/k)$  expected greedy routing time. Our constructions will generate close to  $\log n$  edges per vertex in expectation<sup>1</sup>, and thus have routing time  $O(\log n)$ . They remind most of previously explored finite long-range percolation models, for which the diameter is known to be  $O(\log n/\log \log n)$  [5].

### 3 The Independent Interest Model

We start by introducing a conceptually simple model. Compared to our main model below, it is not a particularly interesting model of networks dynamics, and not particularly realistic, but serves to illustrate the reasoning we will use later.

Let  $X_1, X_2, \dots, X_n$  be  $n$  random variables drawn from an exchangeable joint distribution such that  $\mathbf{P}(X_i = X_j) = 0$  for  $i \neq j$ . It is well known in this case that the probability that for any  $k$ ,  $\mathbf{P}(X_k \geq X_j \text{ for all } j < k) = 1/k$ , and that this event is independent for each  $k$ . This fact, combined with (2) motivates the following graph model

**Definition 3.1.** (The Independent Interest Graph) *Let  $G = (V, E)$  be a graph, and for  $x, y \in V$  let  $d(x, y)$  be the graph (geodesic) distance between them.*

*Create the long range links as follows: For each vertex, independently create an exchangeable sequence of random variables  $(X_y^x)_{y \in V}$ . Then add an edge from  $x$  to  $y$  if:*

$$X_y^x \geq X_z^x \text{ for all } z \neq x : d(x, z) < d(x, y).$$

In the social network metaphor, each  $X_y^x$  in the construction above can be seen as  $x$ 's interest in  $y$ , and the construction simply means that  $x$  befriends each  $y$  who is more interesting to him than any closer person. In other words, starting from his own position,  $x$  searches outwards for friends, befriending each new person he meets if that person is more interesting to him than the people he already knows. In reality, of course, it is unlikely that the interest levels  $X_y^x$  would be independent for each  $x$  and  $y$  – in particular, one would expect a high correlation between  $X_y^x$  and  $X_x^y$ . This fact will inspire our later models below.

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<sup>1</sup>A degree going to infinity may seem unrealistic in terms of social networks, but note that  $\log(6 \text{ billion}) \approx 22.5$  which is probably considerably less than the average number of acquaintances a person has in the real world for most definitions of the word “acquaintance”. Our models can be given bounded degree by simply thinning the edges (removing each edge independently with probability  $1 - 1/\log(n)$ ).

That the independent interest graph is navigable in fact follows from the observations above and previous results, but for illustration we will give a direct proof here.

**Theorem 3.2.** *For any family of connected graphs with bounded doubling dimension, the expected greedy path between any two vertices has expected length  $O(\log n)$ , where  $n$  is the size of the graph.*

*Proof.* Let  $z \in V$  be the target vertex. We follow the standard method: divide  $V$  into  $O(\log n)$  phases, with the  $i$ -th phase defined as the set of vertices  $x$  such that  $2^{i-1} < |x - z| \leq 2^i$ . At a vertex  $x$  in the  $i$ -th phase, for  $i \geq \log \log n$ , let  $A$  be the event that  $x$  has a shortcut to a lower phase, that is

$$A = \{x \rightsquigarrow y : y \in B_{2^{i-1}}(z)\} = \{x \rightsquigarrow B_{2^{i-1}}(z)\}.$$

Let

$$w = \operatorname{argmax}_{v \in B_{(3/2)2^i}(z)} (X_v^x)$$

By construction,  $x$  has a link to  $w$ , so  $A$  will occur if  $w \in B_{2^{i-1}}(z) \subset B_{(3/2)2^i}(x)$ . That the family has bounded doubling dimension thus means there is a constant  $c$  such that  $B_{(3/2)2^i}(x)/B_{2^{i-1}}(z) \leq c^2$ . Since each vertex in the larger ball is equally likely to be the most interesting.

$$\mathbf{P}(A) \geq \frac{1}{c^2}$$

independent of  $n$  and  $i$ . If  $A$  does not occur, then in the next step we are by necessity not further from  $z$  (nearest neighbors in base graph are always connected), and because the edges are chosen independently,  $A$  occurs at the new vertex with at least the same probability. Therefore the expected number of steps until  $A$  occurs, an upper bound on the number steps in a phase, is at most  $c^2$ .

For each sufficiently big phase, we thus have a constant bound on the expected number of steps. Since the destination of the edges at each vertex are independent of the previous path taken by the query, it follows that the expected number of steps in such phases is at most the sum over all of them, which is  $O(\log n)$ . Only  $O(\log n)$  points in smaller phases remain, so the result holds.  $\square$

## 4 The Double Clustering Model

Our main model of interest is conceptually similar to the independent interest model of the last section, but rather than letting each vertex' interest in each other vertex be an independent random variable, we let each vertex

also live in a second space, and let the interest between two vertices be their proximity in that space. In a social network, this would can be represented by each individual not only living somewhere in the physical world, but also having some position in a less clearly defined “space of interests” (his job, activities, etc.). People who live close to one another tend to become acquainted by “default”, while people befriend those far away only if they interests that agree to some extent.

In the constructed graph, each vertex is thus connected to every other vertex that is at least as “interesting” as any other that is at most as “far away”. Formally, let a distance function be a real valued kernel  $d(x, y)$  such that  $d(x, x) = 0$  and  $d(x, y) + d(y, z) \geq d(x, z)$  but which is not necessarily symmetric. The most general definition of such a graph is then:

**Definition 4.1.** (The Double Clustering Graph) *Let  $\{x_i\}_{i=1}^n$  and  $\{y_i\}_{i=1}^n$  be set two sets of points in possibly different spaces  $M_1$  and  $M_2$  with distance functions  $d_1$  and  $d_2$  respectively. The graph  $G = (V, E)$  is constructed as follows:*

- $V = \{1, 2, \dots, n\}$ .
- $(i, j) \in E$  if for all  $k \in V$ ,  $k \neq i, j$ :

$$d_1(x_i, x_k) < d_1(x_i, x_j) \Rightarrow d_2(y_i, y_k) \geq d_2(y_i, y_j)$$

Note that the two sequences are symmetric in the definition, and that  $G$  contains a nearest neighbor graph for both point sets. If, in particular, we let the  $x_i$  and  $y_i$  be the vertices of two graphs  $G_1$  and  $G_2$ , letting  $d_1$  and  $d_2$  be graph distance, we may see the construction as an augmentation of either one to create a denser graph.

Since we are interested in probabilistic models, we want to let  $(x_i)$  and  $(y_j)$  be random points. One way of doing this is to let  $\pi$  be a random permutation of  $[n]$ , and then letting  $y_i = x_{\pi(i)}$ . In the graph case, this corresponds to:

**Definition 4.2.** (Random Double Clustering Graph) *For a vertex set  $V$ , let  $G_1 = (V, E_1)$  and  $G_2 = (V, E_2)$  be a given graphs. Let  $\pi$  be a random permutation of  $V$ , and construct  $G' = (V, E')$  by letting  $(u, v) \in E'$  if for all  $w \in V$ ,  $w \neq u, v$ :*

$$d_1(u, w) < d_1(u, v) \Rightarrow d_2(\pi(u), \pi(w)) \geq d_2(\pi(u), \pi(v))$$

where  $d_1$  and  $d_2$  graph distances in  $G_1$  and  $G_2$  respectively.

Note that every edge added in the construction has a direction, though in many cases (such as nearest neighbors in  $G_1$  and  $G_2$ ) edges in both directions

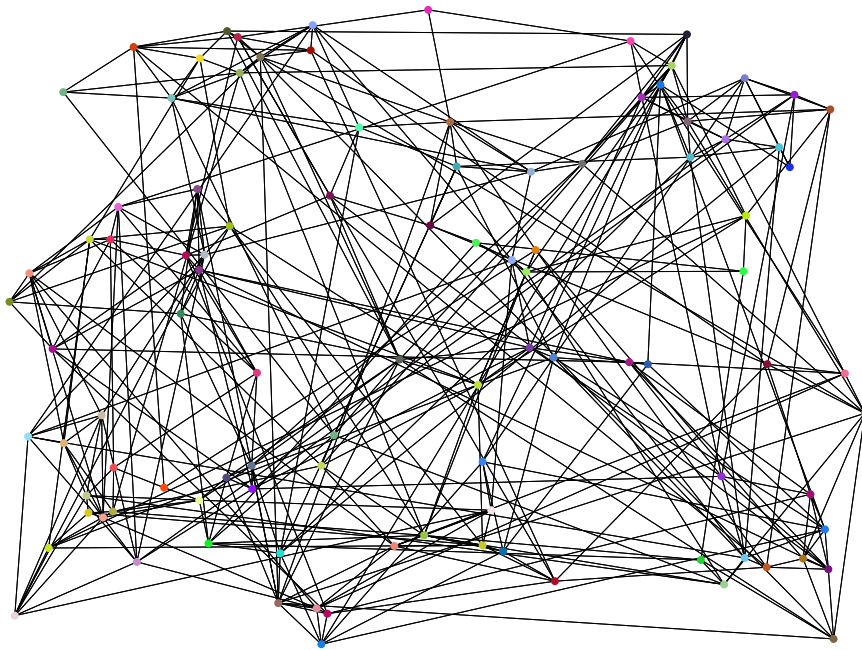


Figure 1: A double clustering graph of 100 vertices. Each vertex has a random position in a two-dimensional physical space  $([0, 1.33] \times [0, 1])$ , as well as a in a three-dimensional color space (RGB)  $([0, 1]^3)$ , both using Euclidean distance.



will be included. One may choose to see the resulting graph as undirected by simply removing directionality and duplicated edges. For the sake of bounding the routing time, it is advantageous to preserve directionality and route using only outgoing edges.

In light of this, and before proceeding to analysis, we note that the construction  $G'$  works equally well if  $G_1$  and  $G_2$  are directed graphs.

## 5 Analysis of Double Clustering

We will analyze special cases of Definition 4.2. We start by proving that greedy routing takes only  $O(\log n)$  steps in expectation when we construct a double clustering graph using two directed cycles. Augmenting a directed cycle is the most basic form of Kleinberg type navigability, and has been extensively investigated in the case of independent augmentation (see e.g. [2]), but of course is not a good model for most real world scenarios.

More general models, in particular where the first space is not directed, are more complicated. In this case the probability of finding a link that halves the distance to the destination is not independent of the previous route taken. This can be seen in the simulations below, where double clustering graphs have slightly longer greedy paths than the equivalent independent interest graphs, though seemingly only by a constant. We attempt an analysis of one class of such models, where the first graph may take a more general form, and the second is an *undirected* cycle (in particular, this includes the case of two undirected cycles), but to do so we are forced to modify the routing used somewhat. The resulting algorithm is still a form of decentralized routing by Kleinberg's definition. Using this, we are able to show that routing takes a polylogarithmic number of steps, a somewhat worse bound than what we expect is true.

We conjecture that double clustering can be applied to just about any graph (see the conclusion), but can not yet prove it.

### 5.1 Two Directed Cycles

Let  $G_1$  and  $G_2$  in Definition 4.2 be two cycles of  $n$  points, that is the directed graphs with vertex set  $V_1 = V_2 = \{0, 1, \dots, n-1\}$  and both  $E_i$  containing an edge from  $u$  to  $u+1$  (modulo  $n$ ) for each  $u \in V_i$ . We will refer to this special case as the Double Cycle Graph. It constitutes the simplest case of double clustering.

Below,  $d(x, y) = y - x \bmod n$  will be graph distance in the cycles, and  $d_\pi$  will be corresponding distance function on the permuted cycle ( $d_\pi(x, y) =$

$d(\pi(x), \pi(y))$ ). We will discuss greedy routing using  $d$ , but by symmetry the same results hold for  $d_\pi$ .

Note from the definition that  $G$  contains a link to the point  $y$  such that  $d(x, y) = 1$  (the next vertex in the cycle) and the point  $z$  such that  $d_\pi(x, z) = 1$  (the next vertex in the permuted cycle).

Addition of vertex values below is always modulo  $n$ , but the notation is suppressed for readability.

**Lemma 5.1.** *For  $w \neq z \in V$ , let  $w' \in V$  be the vertex that  $w$  routes to when  $d$ -greedy routing for  $z$ . Then:*

- $w'$  lies inclusively between  $w + 1$  and  $z$  in the cycle (that is  $d(w', z) < d(w, z)$ ).
- $w'$  lies inclusively between  $w+1$  and  $z$  in the permuted cycle ( $d_\pi(w', z) < d_\pi(w, z)$ ).

*Proof.* The first statement is obvious from the definition of greedy routing, and the fact that  $w \rightsquigarrow w + 1$  so there is always a choice which approaches  $z$ .

To prove the second statement, assume that  $w'$  is not between  $w + 1$  and  $z$  in the permuted cycle. This means that  $d_\pi(w, z) < d_\pi(w, w')$ . Let  $A$  be the set of points inclusively between  $w' + 1$  and  $z$  (that is  $A = \{w' + 1, w' + 2, \dots, z\}$ ). Define  $q$  as the first point in  $A$  such that  $d_\pi(w, q) < d_\pi(w, w')$ , noting that at least one such point,  $z$ , exists. But by construction, and since  $w \rightsquigarrow w'$ ,  $q$  must be closer to  $w$  in the permuted cycle than any vertex between  $w$  and itself, and thus  $w \rightsquigarrow q$ . But if this were the case,  $w$  would have routed to  $q$  and not  $w'$ , which is a contradiction.  $\square$

**Corollary 5.2.** *For any permutation  $\pi$ , a  $d$ -greedy path from any vertex  $y$  to any other  $z$  in the double cycle monotonically approaches  $z$  in  $d_\pi$ , likewise a  $d_\pi$ -greedy path monotonically approaches  $z$  in  $d$ .*

In light of the corollary, it might seem that greedy routing with respect to  $d$  and  $d_\pi$  would produce the same paths. In fact, this is not the case, which we prove as an aside:

**Lemma 5.3.** *There exists a permutation  $\pi$  such that greedy routing from some vertex  $y$  to some vertex  $z$  with respect to  $d$  and  $d_\pi$  produces different paths.*

*Proof.* Let  $\pi$ ,  $y$  and  $z$  be such that there are exactly two vertices  $x_1$  and  $x_2$  that lie between  $y$  and  $z$  in the cycle ( $d(y, z) > d(x_1, z) > d(x_2, z)$ ), and also lie between  $y$  and  $z$  in the permuted cycle. Let  $x_1, x_2$  appear in the opposite order the permuted cycle ( $d(y, z) > d_\pi(x_2, z) > d_\pi(x_1, z)$ ).

Note that by construction,  $y$  will have edges to both  $x_1$  and  $x_2$  in the double cycle graph, because  $x_1$  is closer in the  $d_\pi$  than any  $d$  closer point to  $y$ , and likewise for  $x_2$  (in particular, it is closer to  $y$  than  $x_1$  in  $d_\pi$ ). However, when greedy routing with respect to  $d$  for  $z$ ,  $y$  will choose  $x_2$ , while when greedy routing with respect to  $d_\pi$ , it will choose  $x_1$ .  $\square$

Marginally, under a uniform random choice of  $\pi$ , the probability that  $x \rightsquigarrow y$  in the double cycle model is exactly  $1/d(x, y)$  as it should be for navigability. However, like in all the double clustering graphs, the random edges are not formed independently, so the situation is different from previous results. We will see, however, that in the case of the double cycle, the monotonicity of the routing path also in  $d_\pi$ , as proved above, makes the routing events independent (in a sense which will be shown precisely below): the knowledge provided by previous routing steps is always “behind us” in the permuted cycle.

**Theorem 5.4.** *For any two points  $y, z \in [n]$ , the greedy path from  $y$  to  $z$  in the double cycle graph formed by a uniformly random permutation  $\pi$  has expected length  $O(\log n)$ .*

*Proof.* The proof method is the same as in Theorem 3.2, thus we will consider starting in a point  $x$  such that  $r > d(x, z) \geq r/2$  and bound the expected number of steps (conditioned on the earlier path) until the route is within  $r/2$  of  $z$ .

Divide the vertices between  $x$  and  $z$  in the cycle into two equal sized sets  $R$  and  $H$ , so that if  $d(x, z)$  is odd

$$R = \{x + 1, x + 2, \dots, x + \frac{d(x, z) + 1}{2}\}$$

$$H = \{x + \frac{d(x, z) + 1}{2} + 1, \dots, z - 1, z\}.$$

If  $d(x, z)$  is even, we let  $R$  end at  $x + (d(x, z)/2)$  and  $H$  go from there to  $z - 1$  so that  $R$  and  $H$  retain the same size.

Note that if  $x \rightsquigarrow H$ , then we can route to a point with distance to  $z$  less than  $r/2$ , and that

$$\mathbf{P}(x \rightsquigarrow H) = \mathbf{P}(d_\pi(x, H) < d_\pi(x, R)) = 1/2$$

where  $d_\pi(x, S)$  means the minimal distance from  $x$  to any point in the set  $S$ .

Let  $A$  be the event that  $d_\pi(x, H) < d_\pi(x, R)$ , and  $B$  be the event that before reaching  $x$  we greedy routed along the path

$$x_1 \rightsquigarrow x_2 \rightsquigarrow \dots \rightsquigarrow x_k \rightsquigarrow x$$

for some  $k$  and sequence of vertices where  $d(x_i, z) < d(x, z)$ . We will show that  $\mathbf{P}(B \cap A) = \mathbf{P}(B \cap A^c)$ , which (since  $P(A) = P(A^c)$ ) implies that  $\mathbf{P}(B|A) = \mathbf{P}(B|A^c)$  and thus that  $A$  and  $B$  are independent.

To do this, we define a bijection between the set of permutations  $B \cap A$  and  $B \cap A^c$ . For a given  $\pi \in B \cap A$ , let  $\pi'$  be  $\pi$  composed with a permutation that flips the positions of the elements in  $R$  and  $H$ . Clearly, if  $\pi \in A$ , then  $\pi' \in A^c$ .

By Corollary 5.2  $d_\pi(x_i, z) > d_\pi(x, z)$  for all the  $x_i$  in the definition of  $B$ . This means that all the vertices in  $R \cup H$  are further from each  $x_i$  than  $x$  in both  $d$  and  $d_\pi$ . Thus the internal order of vertices in  $R \cup H$  can not affect the edges of the  $x_i$ , and if  $\pi \in B$ , then  $\pi' \in B$  as well. It follows that  $|B \cap A| = |B \cap A^c|$ , whence

$$\mathbf{P}(A|B) = \mathbf{P}(A) = 1/2$$

for any  $B$  defined as above. At each vertex we reach at distance between  $r$  and  $r/2$  to  $z$ , the probability of having a link to a vertex with distance less than  $r/2$  is thus greater than  $1/2$  regardless of which vertices we visited previously. The result now follows as in Theorem 3.2.  $\square$

## 5.2 Bounded Doubling Dimension and an Undirected Cycle

In this section, we let  $G_1$  belong to a more general family meeting the criteria of Definition 2.1, and we let  $G_2$  be an undirected cycle (a one-dimensional toric grid). Like in previous cases, we shall bound the expected number of steps that it takes to halve the distance to the destination: however, unlike in previous cases, the event of halving the distance in each step of greedy routing is not independent of the previous path.

In order to control the dependencies between the edges encountered at each step, we introduce a modified routing algorithm we call *half-greedy routing*. When routing for a vertex  $z$  and currently at  $x$ , we examine each of  $x$ 's neighbors in the double clustering graph  $G$ . If any neighbor  $w$  is such that  $d_1(x, z) > 2d_1(w, z)$ , then  $w$  is chosen for the next step. If no such  $w$  is found,  $x$  routes to a neighbor  $w'$  in  $G_1$  such that  $d_1(w', z) = d_1(x, z) - 1$  (choosing from all possible such  $w'$  by some deterministic rule).

Half-greedy routing thus either takes a “very big step”, which immediately halves the distance, or a very small step to the next vertex in  $G_1$ . Intuitively, one may imagine this as a participant in a Milgram style experiment only bothering to send the letter by post if he knows somebody very suitable, and otherwise just giving it directly to one of his neighbors. The analytical advantage of this approach is that while subsequent vertices reached by a greedy route do not have independent positions in  $G_2$ , neighbors in  $G_1$  (nearly) do. The navigability result thus follows from this lemma:

**Lemma 5.5.** *Let  $\pi$  be a random permutation of  $[n]$  and  $d_\pi$  be circular distance under this permutation. That is, for  $x, y \in [n]$*

$$d_\pi(x, y) = \min(|\pi(x) - \pi(y)|, n - |\pi(x) - \pi(y)|)$$

*Let  $A$  and  $B$  be disjoint subsets  $[n]$ , such that  $|A| = k$  and  $|B| \geq qk$  for some  $q > 0$ . The elements of  $A$  are enumerated  $a_1, a_2, a_3, \dots, a_k$ . Define a random variable  $\tau$  by*

$$\tau = \min(t \geq 0 : d_\pi(a_t, A \setminus \{a_t\}) \geq d_\pi(a_t, B))$$

*or  $\tau = k$  if this never occurs. Then, for  $t < k/5$*

$$\mathbf{P}(\tau \geq t) \leq e^{-mt}$$

*where  $m = m(q) < \infty$ , a constant independent of  $n$  and  $k$ .*

We will establish this lemma below. First we show how it leads to the desired result.

**Theorem 5.6.** *In Definition 4.2 let  $G_1$  be a connected graph from a family with bounded doubling dimension, and  $G_2$  be an undirected cycle. Then then path though  $G$  between any two vertices  $x$  and  $z$  when half-greedy routing with respect to  $d_1$  has expected length  $O(\log^2 n)$ .*

*Proof.* Let  $T$  be time it takes for half-greedy routing between any two vertices. We will establish the stronger fact that for  $n$  sufficiently large and a constant  $h$

$$\mathbf{P}(T \geq h(\log n)^2) \leq \frac{\log_2 n}{n}. \quad (3)$$

It then follows that

$$\begin{aligned} \mathbf{E}[T] &\leq h(\log n)^2 \left(1 - \frac{\log_2 n}{n}\right) + n \frac{\log_2 n}{n} \\ &= O(\log^2 n). \end{aligned}$$

Fix a destination  $z$ , and let the phases be as in the proof of Theorem 3.2. Consider the  $i$ -th phase (the set of vertices  $x$  such that  $2^{i-1} < d_1(x, z) \leq 2^i$ ), where  $i$  is such that the phase is “big”, meaning it contains more than  $\log^2 n$  vertices. We let  $A$  and  $B$  from Lemma 5.5 be defined by

$$B = B_{2^{i-2}}(z)$$

and

$$A = B_{5(2^{i-1})}(z) \setminus B$$

where the  $B_r(z)$  are balls with respect to  $d_1$ . We note that (distance below always means  $d_1$  except where otherwise noted):

1. Each vertex in the  $i$ -th phase belongs to  $A$ .
2. All vertices in  $B$  are within distance  $3(2^{i-1})$  of any vertex in the  $i$ -th phase.
3. Every vertex within distance  $3(2^{i-1})$  of a vertex in the  $i$ -th phase is in  $A \cup B$ .

Together, these three facts mean that if a vertex  $x$  in  $i$ -th phase has a randomly assigned position in  $G_2$  (the cycle) which is at least as close (with respect to the permuted positions in  $G_2$ ) to a vertex in  $B$  as any vertex other than itself in  $A$ , the resulting double clustering graph  $G$  will have an edge from  $x$  into  $B$ .

Now consider half-greedy routing starting from a vertex  $x$  in the  $i$ -th phase. Let the enumeration of  $A$  be so that  $a_1 = x$  and each subsequent  $a_j$ , for  $j$  up to some  $\ell$ , is the vertex where  $a_{j-1}$  would route if a “very big step” was not found.  $a_\ell$  is the first vertex encountered so that it has a  $G_1$  neighbor in a lower phase, after this we may order the elements of  $A$  as we wish.

Since each vertex in  $B$  is less than half as far from  $z$  as those in phase  $i$ , the random variable  $\tau$  from Lemma 5.5 thus dominates the time we spend in the  $i$ -th phase after starting from a given vertex.

Let  $b = 2/m$ , where  $m$  is the constant from Lemma 5.5 with  $q = 1/c^4 \geq |B|/|A|$  because of the bounded doubling dimension. Note that  $q$ , and thus  $m$  and  $b$ , are independent of which phase we are in. Let  $E_x^i$  be the event that we spend more than  $b \log n$  steps in the  $i$ -th phase after starting from a vertex  $x$  in the phase. Lemma 5.5 and the argument above gives

$$\mathbf{P}(E_x^i) \leq \frac{1}{n^2}.$$

Since the probability is simply uniform measure of permutations of  $[n]$ , this means that starting for any given vertex  $x$  in the phase, routing to the next phase will take more than  $b \log n$  steps in less than  $1/n^2$  of all the permutations. Since the graph is dependent, where we enter the phase may depend on the permutation, but the very worst case scenario is that we always enter the phase at the vertex where it will take the most steps to route to the next. Let  $E^i$  be the event that starting from *any* vertex in the  $i$ -th phase, we spend more than  $b \log n$  steps in the phase.

$$\begin{aligned} \mathbf{P}(E_i) &= \mathbf{P}(\cup_{i\text{-th phase}} E_x^i) \\ &\leq \sum_{i\text{-th phase}} \mathbf{P}(E_x^i) \leq \frac{1}{n} \end{aligned}$$

where the last inequality holds because every phase trivially contains at most  $n$  vertices.

There are at most  $\log_2 n$  “big” phases, so the probability of spending more than  $b \log n$  in any of them is less than  $\log_2 n/n$  by another union bound. Since the number of vertices in the “small” phases is less than  $4 \log^2 n$ , (3) follows with  $h = b/\log(2)$ .  $\square$

The remainder of this section is a proof of Lemma 5.5. In order to establish the Lemma, we will make use of something we call the *toy train track* construction of a random permutation. We equate each vertex on the cycle with a curved segments of track in toy train set. These segments can be attached to each other to make longer sections<sup>2</sup>, and when all the  $n$  segments are attached they form a complete circle. All the pieces start in a bin, and are either red (corresponding to vertices in  $A$ ), blue (corresponding to vertices in  $B$ ), or gray (corresponding to vertices in neither set). We build the random circular track, starting as follows:

1. We pick up the segment of track corresponding to  $a_1$  from bin, this is our current section.
2. Uniformly select from the remaining pieces a segment  $x$  to attach clockwise from the current section, and then another segment  $y$  to attach counterclockwise from the section.
3. As long as neither the  $x$  nor  $y$  picked up in the last step is a blue or red piece, continue we draw two new pieces to attach to the section.

This continues until a red or blue segment has been attached at one or both ends of the section. At this time the first construction stage is completed, and we *put the constructed section of track back in the bin* together with the other pieces. If at least one end was blue, then the building phase terminates.

If no blue piece was found, we start the second construction stage, we try to take out  $a_2$  from the bin. If  $a_2$  cannot be found on its own (it was part of the previous section), then the stage ends immediately. If it is found, then we proceed to build a new section starting from it as in the first stage, but this time we stop whenever a blue segment, a red segment, or the previously constructed section of track is attached to  $a_2$ ’s section. At the end of stage two, we put  $a_2$ ’s section back in the bin as before (if one was built), and, unless a blue piece was found, continue to stage three, which we complete in a similar manner.

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<sup>2</sup>Our chosen vocabulary is to consistently use *segment* for each element, and *section* for connected collections of segments.

If at any time all the pieces have been added to one section the building phase terminates, and likewise if we run out of red pieces to start from. When the building phase has terminated, we attach all the sections and segments in the bin in a random permutation (draw one at a time, and attach clockwise from the last) to form a completed circle.

Let  $X$  be the number of construction stages. We make three claims about this construction which together establish Lemma 5.5:

1. The circle of track segments created is a uniformly random circular permutation.
2.  $X \geq \tau$  (as defined above) for the corresponding permutation.
3. For  $t < k/5$ ,  $\mathbf{P}(X \geq t) \leq e^{-mx}$ , where  $m$  depends on  $q$  but not  $k$  and  $n$ .

*Proof of Claim 1:* This follows from the conditional distribution of random permutations. If one conditions on two segments  $s_1$  and  $s_2$  being next to each other, then resulting distribution is a random permutation of the remaining segments, with the  $s_1s_2$  section uniformly inserted. This is equivalent to the returning of the section to the bin. Likewise, another section  $s_3s_4$  would simply be uniformly inserted again. The claim follows from a series of such arguments.

□

*Proof of Claim 2:* This is almost immediate. If we encounter a blue piece during construction stage  $i$ , then all the segments closer to  $a_i$  than that piece were gray, hence  $d_\pi(a_i, A \setminus \{a_i\}) \geq d_\pi(a_i, B)$ . If we don't find a blue piece in any construction stage, then  $X = k$  which is an upper bound on  $\tau$ .

□

*Proof of Claim 3:* Let  $E_i$  be the event that we encounter a blue piece in the  $i$ -th construction stage.  $X$  is  $\min(i : E_i \text{ occurs})$  or  $k$  if this is undefined. In the first stage, there are  $k + qk$  pieces for which we terminate, and  $qk$  are blue, so  $\mathbf{P}(E_1) \geq q/(1 + q) =: p$  (in fact greater).

Conditioned on  $E_1$  not occurring, we let  $e_1$  and  $e_2$  be the two end pieces, and note that

$$\mathbf{P}(E_2 | E_1^c) = \mathbf{P}(E_2 | E_1^c \text{ and } a_2 \neq e_1, e_2) \mathbf{P}(a_2 \neq e_1, e_2 | E_1^c).$$



If  $a_2 \neq e_1$  or  $e_2$ , then second construction stage could proceed. However, since  $E_1$  did not occur, this means that we removed at least two red segments from the bin, and added only one new terminating section. Thus:

$$\mathbf{P}(E_2 | E_1^c \text{ and } a_2 \neq e_1, e_2) \geq \mathbf{P}(E_1) \geq p.$$

We now have to lower bound  $\mathbf{P}(a_2 \neq e_1, e_2 | E_1^c)$ . The worst case is that both  $e_1$  and  $e_2$  are red, in which case we drew 2 red segments out of  $k - 1$  possible.

$$\mathbf{P}(a_2 \neq e_1, e_2 | E_1^c) \geq 1 - \frac{2}{k-1}$$

Using similar arguments (and rather conservative estimates), it follows that for  $i \leq k/3$ .

$$\mathbf{P}(E_i | E_1^c, E_2^c, \dots, E_{i-1}^c) \geq p \left( 1 - \frac{2(i-1)}{k-(i-1)} \right) = p \frac{k-3(i-1)}{k-(i-1)} \quad (4)$$

whence

$$\begin{aligned} \mathbf{P}(X \geq t) &= \prod_{i=1}^t \mathbf{P}(E_i^c | E_1^c, E_2^c, \dots, E_{i-1}^c) \\ &\leq \prod_{i=1}^t \left( 1 - p \frac{k-3(i-1)}{k-(i-1)} \right) \\ &\leq \left( 1 - \frac{p}{2} \right)^t = e^{-mt} \end{aligned}$$

where  $m = -\log \left( 1 - \frac{p}{2} \right)$ .

□

## 6 Simulations

Simulations support the conjecture that double clustering creates navigable graphs over a larger span of structures. In cases where the first graph is not a directed cycle, one can see that double clustering gives slightly worse performance than when the edges are independent, as is expected. However, the simulation data still strongly indicates a logarithmic growth of path-length with the size of the graph.

### 6.1 Combined Greedy Routing

Since the double clustering construction is symmetric, it should create equally navigable networks with regard to both spaces. Thus we can perform greedy

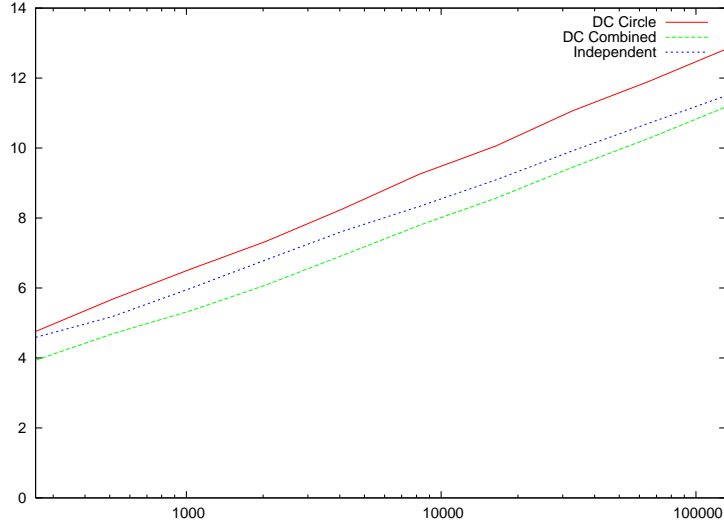


Figure 2: Performance of double clustering graph versus the independent interest model using two undirected cycles.

routing in the double clustering graph with respect to either distance function (which will sometimes lead to different results, see below.)

A direct consequence of this is that we may try to route with respect to both distance functions, using at each step that which seems most profitable. As above, assume that  $z$  is the target of the route.

- At vertex  $x$ , we calculate  $m_1 = d_1(w_1, z)$ , where  $w_1$  is the neighbor of  $x$  which minimizes this. Similarly, calculate  $m_2 = d_2(w_2, z)$ .
- Let  $n_1$  be the number of vertices within  $m_1$  of  $z$  in the first space ( $M_1$ ) - if the space is homogeneous this is the volume of a ball of diameter  $m_1$ . Let  $n_2$  be the equivalent for  $m_2$  and the second space.
- Route to  $w_2$  if  $m_2$  is smaller than  $m_1$ , otherwise  $w_1$ .

We simulate combined routing as well as normal greedy routing for the models below. In these models it seems that the benefit of using this method regains that lost by the dependencies in the double clustering construction: combined greedy paths are shorter than greedy paths in the independent interest model of the same size.

## 6.2 Two Undirected Cycles

The simplest undirected double clustering model is the case of Definition 4.2 where both  $G_1$  and  $G_2$  are undirected cycles. A bound on half-greedy

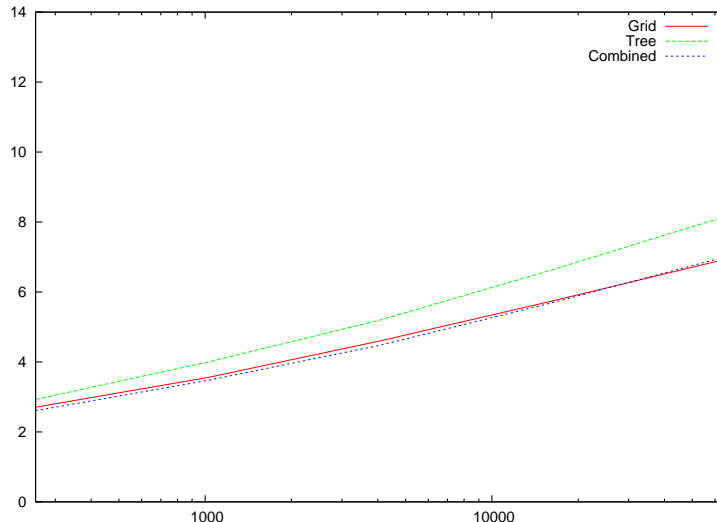


Figure 3: Performance of double clustering using letting the first space (the geography) be a two dimensional grid, and the second a tree with the points as leaves (a categorization).

routing in this model is derived above, but we can simulate also the normal greedy algorithm. The results illustrated in Figure 2 - at all sizes simulated greedy routing with respect to either cycle produces slightly longer paths than the equivalent independent model, while combined greedy routing produces slightly shorter paths. All lines seem to follow strictly logarithmic growth.

### 6.3 A Grid and a Tree

Kleinberg's original work consisted started with a two dimensional grid as the base graph and distance function, inspired, one expects, by the dimensionality, if not population distribution, of the surface of the earth. Later he [14] and Watts et al. [19] proposed equivalent models based on letting vertices have positions at the leaves of a tree. The tree represents a hierarchical model of information, ideas, interests or other characteristics, and the distance function is standard tree distance:  $d(x, y)$  is the depth of smallest subtree containing both. The criteria for navigable augmentation in these cases is consistent with (2).

A natural attempt at a realistic double clustering model is to combine both of Kleinberg's models - we let the first space be a grid, and the second be a hierarchical tree structure (in our case, a binary tree, though any other branching is possible). We note that while the tree distance provides a well defined metric, this space can not be seen as a graph, so this is a sub-model

of Definition 4.1 rather than Definition 4.2. A problem with the more general model is that greedy routing is not necessarily always successful: we may reach a vertex other than the destination with no neighbor which is closer to the destination than itself. This can occur in this model when routing with respect to the tree, or the combined distance, but not in the grid (where links to neighbors in all directions always exist) - in our simulations we simply fail and discard such routes<sup>3</sup>.

Figure 3 shows a simulation of this situation. Routing purely using the tree shows slightly worse performance than routing using the grid, and as such the advantage of the combined model is less than above (for the largest data-point simulated it was, in fact, nonexistent). As expected not all routes were successful - at a network size of  $2^{16}$  about 0.8 of the routes using only the tree, and 0.9 of the routes using the combined routing were successful. Another effect of the tree is that the degree of the double clustering graph is much higher (since many vertices have the same distance, and we only require them to be as close as any previous.)

## 6.4 Continuum Models

Discrete and grid based models cannot realistically describe most naturally occurring networks: especially social networks which are characterized by individuals placed randomly in continuums and often with heterogeneous population density. Continuum models for navigable networks have been explored by Franceschetti and Meester [11] and [6] as well this author [17], and Liben-Nowell et. al [15] has proposed a model based on real data that includes non-uniform Poisson density of positions. Figure 1 shows a simulation a continuum model with 100 vertices.

## 7 Conclusion

We have introduced a new form random graph construction, which when combined with a random permutation of the points used to create the graph gives rise to networks with navigable properties. These graphs are constructed from a single natural principle, and may help explain why networks of this type occur in real world networks.

While we have established navigability under a several cases, the analysis presented here is far from complete. In a sense it is unfortunate that we

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<sup>3</sup>When routing for  $z$ , we do allow  $x$  to route to a vertex at the same distance as itself if no better choice, but (so as to not cause loops) we forbid routing to a vertex already in the path). This is important since tree distance has the property that there are a very large number of vertices at the same distance from any other, a large majority of the routes fail when routing for tree distance if this is not allowed.

are able to analyze an unrealistic model (the directed cycle) for an intuitive clear routing principle, while the proof for the more realistic model requires somewhat contrived routing. Theorem 5.6 also has an extra  $\log n$  multiple included for technical reasons in the proof: we believe strongly that neither this term (the actual bound is  $O(\log n)$ ) nor the use of half-greedy routing is actually necessary. In fact, based on the absence of any opposing evidence in simulations or otherwise, we believe

**Conjecture 7.1.** *Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be two families of graphs with bounded doubling dimension (not necessarily with the same constants). For any two graphs  $G_1 \in \mathcal{F}_1$  and  $G_2 \in \mathcal{F}_2$  of size  $n$ , the doubling clustering graph from Definition 4.2, allows greedy routing in  $O(\log n)$  expected steps.*

Proving this in general is difficult since the structure of the two base graphs control the dependence between the edges in the construction. We are however hopeful that progress can be made in this direction. Making rigorous stronger conjectures about Definition 4.1 is also difficult since monotonic greedy paths between vertices may not always exist, but we believe that the resulting graph will be navigable whenever such augmentation is possible.

Beyond this, the double clustering graph, as a new form of graph construction, has not been analyzed for questions other than navigability. Questions such as connectivity, diameter, and edge length remain open in some or all cases. And, finally, the question of how well double clustering actually matches the real world has not been investigated.

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